Lecture 10: Basic Electromagnetic Theory and Plasma Physics

In a similar way as thermodynamics and thermochemistry are the core disciplines required to describe the fundamentals of chemical propulsion, electromagnetic theory and its extension to plasma physics are essential to understand the way in which most electric propulsion devices work. In this lecture we will review the basic formulation of electromagnetism, leading to Maxwells equations and will introduce the fundamentals of plasma physics.

The theory behind electricity and magnetism (and the unification of these two, once separate forces) is one of the major triumphs in the history of science. Its level of accuracy is extraordinary. Remarkable as it is, this fact is not surprising, since the origin of the theoretical description of electric and magnetic fields is based on five well-identified experimental observations.

The **first** of these experimental observations led eventually to Coulombs law, which says that electrically charged materials will exert forces on each other that vary with the inverse of the square of distance between the charges. This observation is then characterized by the force a mass of charge feels,

$$\vec{F} = q\vec{E} = \frac{q^2/r^2}{4\pi\varepsilon_0}\hat{r} \tag{1}$$

when an identical charged mass produces a field $\vec{E}$. According to Helmoltz theorem, a vector is completely determined when both its divergence and curl are known. To find the div and curl of the electric field vector, we first note that,

$$-\frac{d(1/x)}{dx} = \frac{1}{x^2} \quad \rightarrow \quad \vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = -\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r}\right) = -\nabla\phi \tag{2}$$

where ϕ is the electric potential. The curl of the electric field is then,

$$\nabla\times\vec{E} = -\nabla\times\nabla\phi \quad \rightarrow \quad \nabla\times\vec{E} = 0 \tag{3}$$

Now let us integrate the projection of the field over an arbitrary surface S,

$$\int_S \vec{E}\cdot d\vec{S} = \int E\cos\theta\, dS = \int \frac{q\cos\theta}{4\pi\varepsilon_0 r^2}dS = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{\varepsilon_0} \tag{4}$$

where we have used the definition of solid angle $d\Omega = \dfrac{\cos\theta}{r^2}dS$, which integrates to $\int d\Omega = 4\pi$ in all space (even though S is arbitrary, only projections along r are considered, i.e., the projected area is spherical). Now, form the divergence theorem,

$$\int_S \vec{E}\cdot d\vec{S} = \int_V \nabla\cdot\vec{E}dV \tag{5}$$

and since the net charge q in general could arise from a distribution, $q = \int_V \rho dV$, we obtain,

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Gauss' law)} \qquad \text{and} \qquad \nabla^2\phi = -\frac{\rho}{\varepsilon_0} \quad \text{(Poisson's equation)} \tag{6}$$

In the **second** experimental observation, a magnetic dipole (any magnet for that matter) tends to align with an externally applied magnetic field after being subjected to a torque,

$$\vec{T} \;=\; \vec{m} \times \vec{B} \tag{7}$$

The **third** observation is that no isolated magnetic charges have been found in nature. The divergence of a field is a measure of the local density of sources of that field. Since there are no magnetic monopoles, we have,

$$\nabla \cdot \vec{B} \;=\; 0 \tag{8}$$

In the **fourth** experimental observation it is found that an electric current generates a magnetic field such that,

$$\oint_C \vec{B} \cdot \vec{dl} \;=\; \mu_0 I \tag{9}$$

Applying Stoke's theorem to this expression,

$$\oint_C \vec{B} \cdot \vec{dl} \;=\; \int_S \nabla \times \vec{B} \cdot d\vec{S} \tag{10}$$

and for a current arising from a distribution $I = \int_S \vec{j} \cdot d\vec{S}$, we obtain,

$$\nabla \times \vec{B} \;=\; \mu_0 \vec{j} \tag{11}$$

So far, the equations above describe static situations. In the **fifth** and final experimental observation, a non-steady coupling between electric and magnetic fields appears. This coupling is known as Faradays law, in which a time variation of magnetic flux induces an electric field,

$$\oint_C \vec{E} \cdot \vec{dl} \;=\; -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} \tag{12}$$

Using Stoke's law, we notice that Eq. (3) is just an incomplete version of,

$$\nabla \times \vec{E} \;=\; -\frac{\partial \vec{B}}{\partial t} \qquad \text{(Faraday's law)} \tag{13}$$

Up until the 1860's, these equations described the experimental reality observed in the laboratory. It was J.C. Maxwell who synthesized them into a well-known set of four equations, appropriately called Maxwell's Equations. The key insight was noticing that the equations above are not consistent with conservation of charges. To see this, take the divergence of Eq. (11),

$$\nabla \cdot \nabla \times \vec{B} \;=\; \mu_0 \nabla \cdot \vec{j} \;=\; 0 \tag{14}$$

But charge continuity requires that,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} \; = \; 0 \tag{15}$$

To fix this contradiction, we take the time derivative of Gauss's law Eq. (6),

$$\frac{\partial \rho}{\partial t} \; = \; \varepsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} \tag{16}$$

and add this to Eq. (14) so that Eq. (15) is satisfied,

$$\nabla \cdot \nabla \times \vec{B} \; = \; \mu_0 \nabla \cdot \vec{j} + \mu_0 \varepsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} \; = \; 0 \tag{17}$$

The conclusion is that Eq. (11) needs to be modified to read,

$$\nabla \times \vec{B} \; = \; \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \; = \; 0 \tag{18}$$

where the second term is known as the displacement current, typically very small in magnitude at low frequencies, but essential in other situations, for example in the propagation of electromagnetic waves.

Maxwell's Equations (6, 8, 13 and 18) describe the way in which electric and magnetic fields behave in vacuum. They require further modification if the situation under consideration contains matter. The atoms in materials react to applied electric and magnetic fields generating fields on their own that in turn modify the magnitude and direction of applied fields. The introduction of these modifications is relatively straightforward and because of its fundamental importance is discussed here.

Start with the electric field. The charge density in Gauss's law Eq. (6) can be decomposed into two parts, one containing the distribution of free charges and a second including the charge density introduced by the alignment of dipoles in materials when exposed to an electric field,

$$\nabla \cdot \vec{E} \; = \; \frac{\rho_f + \rho_d}{\varepsilon_0} \tag{19}$$

We assume that this dipolar charge density arises from an electric vector field produced by the preferential alignment of dipoles. Clearly, such vector field acts in opposition to the applied field, and is known as the polarization vector,

$$\rho_d \; = \; -\nabla \cdot \vec{P} \tag{20}$$

Substituting into Gausss law gives,

$$\nabla \cdot \vec{D} \; = \; \rho_f \tag{21}$$

where the *electric displacement* vector is,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \qquad \text{or} \qquad \vec{D} = \varepsilon \varepsilon_0 \vec{E} \tag{22}$$

where the second expression was found after expanding the polarization vector in a Taylor series with respect to the electric field. Only linear terms are considered with no permanent polarization, so $\vec{P} \approx \chi_e \vec{E}$. The linear constant is known as the electric susceptibility. The relative permittivity, or dielectric constant of the material, is $\varepsilon = 1 + \chi_e/\varepsilon_0$.

In a similar way, the current density in Ampere's law (ignoring the displacement current, for now) can be decomposed into current carried by free charges and secondary currents induced in the material,

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j}_f + \vec{j}_m \right) \tag{23}$$

In this case, we assume the induced currents are well localized at the atomic level and are produced through the appearance of some magnetization vector defined by,

$$\vec{j}_m = \nabla \times \vec{M} \tag{24}$$

Substituting into Ampere's law results in,

$$\nabla \times \vec{H} = \vec{j}_f \tag{25}$$

where,

$$\vec{H} = \frac{B}{\mu_0} - \vec{M} \qquad \text{or} \qquad \vec{H} = \frac{\vec{B}}{\mu} \tag{26}$$

Once more, the second expression was found when expanding the magnetization vector in a Taylor series in $\vec{H}$ and taking only linear terms with no permanent magnetization, so that $\vec{M} \approx \chi_m \vec{H}$. The constant is known as the magnetic susceptibility and $\mu = \mu_0(1 + \chi_m)$ is the magnetic permeability of the material.

In addition to a set of differential equations, boundary conditions are required to evaluate the fields. The most convenient boundaries are at the interfaces between materials with different magnetic and electric properties. Integrating the static version of Gausss and Amperes laws on a pillbox with vanishingly small volume, but non-zero area at the interface results in a set of four boundary conditions for the fields:

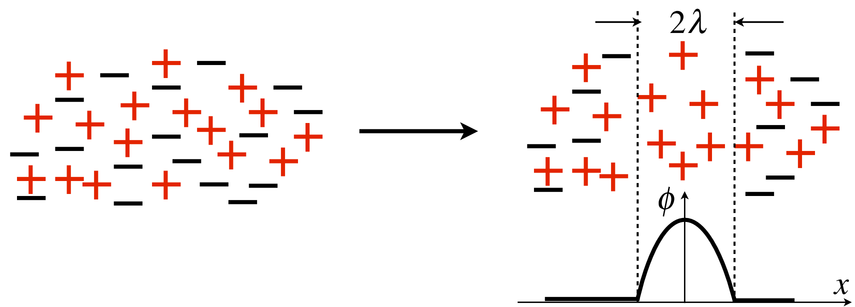
1. The tangential component of the electric field is continuous $E_t^{in} = E_t^{out}$.
2. The normal component of the displacement vector has a jump proportional to the free surface charge at the interface or $D_n^{out} - D_n^{in} = \sigma_f$.
3. The normal component of the magnetic field is constant $B_n^{in} = B_n^{out}$.
4. The tangential component of the magnetic vector field $\vec{H}$ has a discontinuity proportional to the surface current at the interface $H_t^{out} - H_t^{in} = j_f$

Elementary Plasma Physics

In Physics, a plasma is defined as a collection of positively charged ions, negatively charged electrons and neutral species. The main characteristic of the plasma is that, when looked as a macroscopic average is electrically neutral, i.e., same number of positive and negative charges. This neutrality eventually breaks down at the atomic scale. Because of this, it is said that a plasma is, by definition, *quasineutral.* In a general sense, a plasma is like a multi-component fluid, and in many instances its behavior can be described with the same set of tools used to describe regular fluids, with one exception: plasmas react to electric and magnetic fields, and as such we need to consider stresses in addition to the hydrodynamic ones regular fluids experience.

A very important aspect in the description of plasmas pertains to the description of what occurs when perturbed by the application of fields. There are two important quantities that can be derived from such perturbation: the Debye length and the plasma frequency.

To find these, start by considering a 1D plasma in equilibrium in which a force has been applied to separate the electrons in the region $[-\lambda, \lambda]$ as shown below.



The plasma is quasineutral everywhere ($n_e \approx n_i$) except in the region where a net positive charge exists. In general, Poisson's equation Eq. (6) describes the electric potential for a given charge distribution,

$$\nabla^2 \phi \;=\; -\frac{e}{\varepsilon_0}\left(n_i - n_e\right) \tag{27}$$

In the quasineutral region we have Laplace's equation $\nabla^2\phi = 0$ which in 1D integrates easily to $\phi_{out} = Ax + B$. Defining zero potential at $-\lambda$ and λ, we find $A = B = 0$. In the non-neutral region, the ion density equals the plasma density n_e, therefore, the integrals of Poisson's equation result in,

$$\frac{d\phi_{in}}{dx} = -\frac{en_e}{\varepsilon_0}x + C \qquad \text{and} \qquad \phi_{in} = -\frac{en_e}{2\varepsilon_0}x^2 + Cx + D \tag{28}$$

At $x = 0$ we have $\dfrac{d\phi_{in}}{dx} = 0$, then $C = 0$, while $D = \dfrac{en_e}{2\varepsilon_0}\lambda^2$. The potential distribution is,

$$\phi_{in} \;=\; \frac{en_e\lambda^2}{2\varepsilon_0}\left[1 - \left(\frac{x}{\lambda}\right)^2\right] \tag{29}$$

We now have defined a region in a plasma that is non-neutral in which a parabolic electric potential develops. The potential tends to attract electrons back into the region, so for this to be a stable configuration in equilibrium, energy has to be provided to maintain the separation of charges. Let us compute the energy required to move one electron from $x = 0$ to λ under this parabolic potential,

$$W = \int_0^\lambda F dx = \int_0^\lambda eE dx = \int_0^\lambda e\left(-\frac{d\phi}{dx}\right) dx \;=\; \frac{e^2 n_e \lambda^2}{2\varepsilon_0} \tag{30}$$

This energy must be provided by the thermal motion of electrons in the plasma, or $kT_e/2$. From here we solve for $\lambda = \lambda_D$, the Debye length,

$$\lambda_D \;=\; \sqrt{\frac{\varepsilon_0 k T_e}{e^2 n_e}} \tag{31}$$

The Debye length provides an estimate for the size of non-neutral regions in a plasma and is critical to understand the effects of plasmas in contact with materials. For instance, a charged piece of material immersed in a plasma quickly becomes shielded by the production of a Debye layer, effectively nulling the fields produced by the charges on the material.

In the previous statement, *quickly* is quite a vague term. The rate at which charges redistribute in a plasma are also of critical importance, specially when dealing with high frequency phenomena. Imagine the same situation described in the drawing above, but this time assume the electron cloud is mechanically shifted to produce the charge separation. The restoring force per unit volume is $f = -en_eE$, so the equation of motion for electrons,

$$m_e n_e \frac{d^2x}{dt^2} + en_eE = 0 \qquad \rightarrow \qquad \frac{d^2x}{dt^2} + \omega_p^2 x = 0 \tag{32}$$

where,

$$\omega_p \;=\; \sqrt{\frac{e^2 n_e}{m_e \varepsilon_0}} \tag{33}$$

is the plasma frequency, or the rate at which electrons oscillate about their equilibrium positions with respect to a practically immobile ion background. The plasma frequency is very relevant in the propagation of electromagnetic waves. It is intuitively clear that waves with frequencies lower than the plasma frequency will not propagate as their fields are shielded by the plasma, while waves of higher frequencies will propagate as electrons in the plasma do not have enough time to react to the varying fields. The plasma frequency is also of fundamental importance to describe the dynamics of Debye layer formation. In fact, taking the product of the Debye length and plasma frequency,

$$\omega_p \lambda_D \;=\; \sqrt{\frac{kT_e}{m_e}} \tag{34}$$

we obtain the mean thermal velocity (speed of sound) of electrons in the plasma.

Plasmas then become very dynamic conductive media that respond to electric and magnetic fields, producing fields of their own which then need to be self-consistently incorporated into the particle dynamics (though Poisson's equation). A detailed description of the plasma dynamics becomes very challenging. Nevertheless, in many cases, plasmas are dilute enough that charged particles behave almost independently of each other. In those cases, a relatively simple description of the motion of individual charges suffices to predict the behavior of the plasma.

The generic dynamic interaction between electromagnetic fields and particles of charge is determined by the Lorentz force,

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad \text{so that} \qquad m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{35}$$

Restricting for now to planar motion (2D) and assuming $\vec{B}$ is perpendicular to this plane, Eq. (35) could be decomposed into two equations, one for each direction in an orthogonal system of coordinates. Alternatively, we could make use of the intrinsic orthogonality of the complex plane to write a single differential equation, such that,

$$m\frac{dw}{dt} = q\left(E - iwB\right) \tag{36}$$

where both w and E are complex quantities. Let us for now assume that $E = 0$. It is easy then to integrate Eq. (36) into,

$$w = w_0 e^{-i\omega_c t} \tag{37}$$

This represents a clockwise rotation of the velocity vector (of constant magnitude w_0). Eq. (37) can be integrated once more to obtain,

$$z = i\frac{w_0}{\omega_c} e^{-i\omega_c t} \tag{38}$$

The position vector z, is also a complex quantity. We observe this vector also rotates clockwise, although 90° out of phase with respect to the velocity vector. This means that the particle performs a circular motion around the magnetic field. A negative particle will gyrate in the opposite direction. The gyration rate ω_c is known as the cyclotron frequency, and the gyration radius,

$$r_L = \frac{w_0}{\omega_c} = \frac{mw_0}{qB} \tag{39}$$

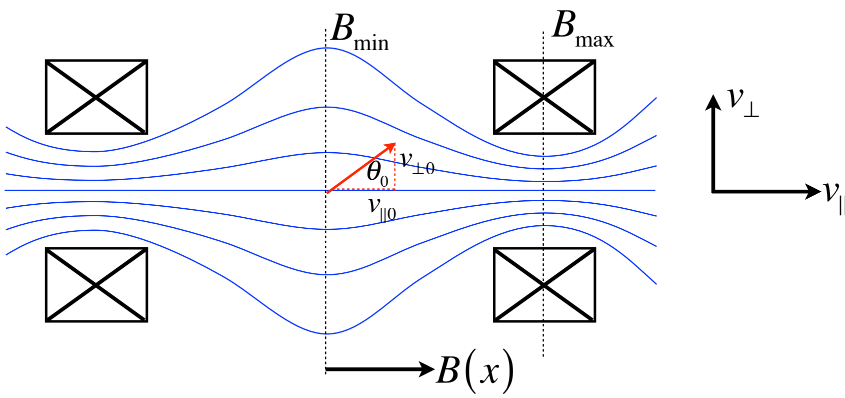
is known as the Larmor radius. The cyclotron motion is also known as the Larmor motion. Whenever a particle family in a plasma (electrons, or different types of ions) perform cyclotron motion, it is said that such charged species is ***magnetized***.

If $E \neq 0$ (but constant), we still have the Larmor solution as the homogeneous part of Eq. (36), but in addition we have a particular solution,

$$w_p = \frac{qE}{im\omega_c} = -i\frac{E}{B} = -i\frac{EB}{B^2} \qquad \rightarrow \qquad \vec{v}_p = \frac{\vec{E} \times \vec{B}}{B^2} \tag{40}$$

So the final solution is a cycloid, in which the Larmor motion is superimposed to a guiding-center drift in the $\vec{E} \times \vec{B}$ direction. Not surprisingly this is called the $\vec{E} \times \vec{B}$ drift. Note that the drift is always in the same direction, regardless of the particle charge.

Increasingly complicated cases could be analyzed with the set of tools presented so far. Of particular relevance for some electric propulsion devices are particle drifts associated with gradients of the magnetic field, in particular the motion of charged particles inside a *magnetic mirror*. A schematic of this situation is shown below.



Through a more detailed analysis of magnetized particle motion, it is found that in addition to the conservation of energy (kinetic only in this case, if there is no electric potential),

$$K = \frac{mv_{\perp}^2}{2} + \frac{mv_{\parallel}^2}{2} \tag{41}$$

there is a second motion (or adiabatic) invariant: the magnetic moment. In this case, the magnetic moment is defined as the product of current and area enclosed by the current loop, or $\mu = \frac{1}{2}\int \vec{r} \times \vec{j} dV = IA$. Assuming only one type of particle is magnetized, the current will be given by the cyclotron current and the area will be defined by the Larmor radius so that,

$$I = q\frac{\omega_c}{2\pi} \qquad \text{and} \qquad A = \pi r_L^2 \qquad \rightarrow \qquad \mu = \frac{mv_{\perp}^2}{2B} \tag{42}$$

The magnetic moment then becomes equal to the local value of the perpendicular (to $\vec{B}$) component of the kinetic energy divided by the magnitude of the magnetic field.

Combining Eqs. (41-42) we find,

$$K = \mu B + \frac{mv_{\parallel}^2}{2} \tag{43}$$

Since both K and μ are constant, it is then clear that the parallel velocity will decrease with B and vice versa. In the schematic shown above, particles will slow down as they approach regions at the edge of the device. But according to Eq. (41), as particles slow down in one direction, their velocity should increase in the other direction. This peculiar energy transfer is representative of drifts generated by gradients in the magnetic field.

If particles slow down as they approach the regions of the highest fields, there must be a possibility for them to stop altogether and bounce back. From here that these ∇B devices are also known as magnetic mirrors. Assume that particles stop and bounce at a particular value of the field $B = B_B$ and if $B_0 = B_{min}$, we have,

$$\mu B_0 + \frac{m v_{\parallel_0}^2}{2} \;=\; \mu B_B \tag{44}$$

And from Eq. (42) we also have $\mu = \dfrac{m v_{\perp_0}^2}{2B_0}$. Combining these we find,

$$\left(\frac{v_{\parallel_0}}{v_{\perp_0}}\right)^2 \;=\; \frac{B_B}{B_0} - 1 \tag{45}$$

Defining the pitch angle,

$$\tan\theta_0 \;=\; \frac{v_{\perp_0}}{v_{\parallel_0}} \tag{46}$$

we finally write,

$$\frac{B_B}{B_0} = 1 + \cot^2\theta_0 = \frac{1}{\sin^2\theta_0} \qquad \rightarrow \qquad \theta_0 = \arcsin\sqrt{\frac{B_0}{B_B}} \tag{47}$$

In conclusion, particles with angles lower than θ_0 will continue beyond B_B, while particles with larger pitch will bounce. The lowest possible pitch angle for bouncing is given by the maximum field $B_B = B_{max}$. Magnetic mirrors are important in fusion research as one of several devices able to confine hot plasma in a vessel. The theory also explains how ionospheric charges precipitate at the earth poles, and describes the performance of magnetic nozzles in plasma thrusters.