Lecture 12: Plasmas in Equilibrium

Ionization and Conduction in a High-pressure Plasma

A normal gas at $T < 3000°$K is a good electrical insulator, because there are almost no free electrons in it. For pressures > 0.1 atm, collision among molecule and other particles are frequent enough that we can assume Local Thermodynamic Equilibrium (LTE), and in particular, ionization-recombination reactions are governed by the law of mass action. Consider neutral atoms (n) which ionize singly to ions (i) and electrons (e):

$$n \leftrightarrow e + i \tag{1}$$

One form of the law of mass action (in terms of number densities $n_j = p_j/kT$, where T is the same for all species) is,

$$\frac{n_e n_i}{n_n} = S(T) \tag{2}$$

Where the "Saha function" S is given (according to Statistical Mechanics) as,

$$S(T) = 2\frac{q_i}{q_n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left(-\frac{eV_i}{kT}\right) \tag{3}$$

- Ground state degeneracy of ion ($q_i = 1$ for H+)
- Ground state degeneracy of neutral ($q_n = 2$ for H)
- Mass of electron ($m_e = 9.11 \times 10^{-31}$ kg)
- Boltzmann constant ($k = 1.38 \times 10^{-23}$ J/K)
- Planck's constant ($h = 6.62 \times 10^{-34}$ Js)
- Ionization potential of the atom ($V_i = 13.6$ V for H)

Except for very narrow "sheaths" near walls, plasmas are quasi-neutral:

$$n_e = n_i \tag{4}$$

So that,

$$\frac{n_e^2}{n_n} = S(T) \tag{5}$$

can be used. The ionization fraction is defined as $\alpha = \dfrac{n_e}{n_e + n_n}$.

Given T, this relates n_e to n_n. A second relation is needed and very often it is a specification of the overall pressure,

$$p \;=\; (n_e + n_i + n_n)kT \;=\; (2n_e + n_n)kT \tag{6}$$

Combining (5) and (6),

$$n_e^2 \;=\; S(T)\left(\frac{p}{kT} - 2n_e\right) \;=\; S(T)\,(n - 2n_e)$$

Where $n = p/kT$ is the total number density of all particles.

We then have,

$$n_e^2 + 2Sn_e - Sn \;=\; 0$$

and,

$$n_e \;=\; -S + \sqrt{S^2 + Sn} \;=\; \frac{n}{1 + \sqrt{1 + \frac{n}{S}}} \tag{7}$$

Since S increases very rapidly with T, the limits of (7) are,

$$n_e\,(T \approx 0) \;\to\; \sqrt{Sn} \qquad \text{Weak ionization}$$
$$n_e\,(T \to \infty) \;\to\; \frac{n}{2} \qquad \text{Full ionization}$$

Once an electron population exists, an electric field $\vec{E}$ will drive a current density $\vec{j}$ through the plasma. To understand this quantitatively, consider the momentum balance of a "typical" electron. It sees an electrostatic force,

$$\vec{F}_e \;=\; -e\vec{E} \tag{8}$$

It also sees a "frictional" force due to transfer of momentum each time it collides with some other particle (neutral or ion). Collisions with other electrons are not counted, because the momentum transfer is in that case internal to the electron species. The ions and neutrals are almost at rest compared to the fast-moving electrons, and we define an effective collision as one in which the electron's directed momentum is fully given up. Suppose there are ν_e of these collisions per second (ν_e is the collision frequency per electron). The electron loses momentum at a rate $-m_e\vec{u}\nu_e$, where $\vec{u}$ is the mean directed velocity of electrons, and so,

$$\vec{F}_{friction} \;=\; -m_e\vec{u}\,\nu_e \tag{9}$$

On average,

$$\vec{F}_e + \vec{F}_{friction} \;=\; 0$$

or $m_e\vec{u}\nu_e = -e\vec{E}$, so that,

$$\vec{u} \;=\; -\left(\frac{e}{m_e\nu_e}\right)\vec{E} \tag{10}$$

The group $\mu = \dfrac{e}{m_e\nu_e}$ is the electron "mobility" [(m/s)/(volt/m)]. The current density is the flux of charge due to motion of all charges. If only the electron motion is counted (it dominates in this case),

$$\vec{j} \;=\; -en_e\vec{u} \;=\; \left(\frac{n_e e^2}{m_e\nu_e}\right)\vec{E} \tag{11}$$

The group,

$$\sigma \;=\; \frac{n_e e^2}{m_e\nu_e} \tag{12}$$

is the electrical conductivity of the plasma (Si/m).

Let us consider the collision frequency. Suppose a neutral is regarded as a sphere with a cross-section area Q_{en}.

Electrons moving at random with (thermal) velocity c_e intercept the area Q_{en} at a rate equal to their flux $n_e c_e Q_{en}$. Since a whole range of speeds c_e exists, we use the average value $\bar{c}_e$ for all electrons. But this is for all electrons colliding with one neutral. We are interested in the reverse (all neutrals, one electron), so the part of ν_e due to neutrals should be $n_n\bar{c}_e Q_{en}$. Adding the part due to ions,

$$\nu_e \;=\; n_n\bar{c}_e Q_{en} + n_i\bar{c}_e Q_{ei} \tag{13}$$

$\bar{c}_e$ is very different (usually much larger) than u_e. Most of the thermal motion is fast, but in random directions, so that on average it nearly cancels out. The non-canceling remainder is u_e. Think of a swarm of bees moving furiously back and forth, but moving (as a swarm) slowly.

The number of electrons per unit volume that have a velocity vector $\vec{c}_e$ ending in a "box", $d^3c_e = dc_{e_x}dc_{e_y}dc_{e_z}$ in velocity space is defined as $f_e(\vec{c}_e,\vec{x})d^3c_e$, where $f_e(\vec{c}_e,\vec{x})$ is the **distribution function** of the electrons, which depends (for a given location $\vec{x}$ and time t) on the three components of $\vec{c}_e$. In an equilibrium situation all directions are equally likely (isotropic), so $f_e = f_e(c_e)$ only, and one can show that the form is **Maxwellian**,

$$f_e \;=\; n_e\left(\frac{m_e}{2\pi kT_e}\right)^{3/2}\exp\left(-\frac{m_e c_e^2}{2kT_e}\right) \qquad \text{where} \qquad c_e^2 \;=\; c_{e_x}^2 + c_{e_y}^2 + c_{e_z}^2 \tag{14}$$

With the normalization,

$$\iiint f_e d^3c_e \;=\; n_e$$

The mean velocity is then,

$$\bar{c}_e \;=\; \frac{1}{n_e}\iiint c_e f_e d^3 c_e \;=\; \sqrt{\frac{8kT_e}{\pi m_e}} \qquad (15)$$

For electrons,

$$\bar{c}_e \;=\; 6210\sqrt{T_e} \qquad (\text{m/s with } T_e \text{ in °K}) \qquad (16)$$

Note that if there is current, the distribution cannot be strictly Maxwellian (or even isotropic). But since $u_e \ll c_e$, the mean thermal velocity is very close to Eq. (15) anyway.

Regarding the cross sections Q_{en} and Q_{ei}, they depend on the collision velocity $\bar{c}_e$, especially Q_{ei}. This is because the $e-i$ coulombic interaction is "soft", so a very fast electron can pass nearly undeflected near an ion, whereas a slow one will be strongly deflected. The complete theory yields an expression,

$$Q_{ei} \;\approx\; 2.95\times 10^{-10}\frac{\ln\Lambda}{T_e^2} \qquad (\text{m}^2 \text{ with } T_e \text{ in °K}) \qquad (17)$$

where

$$\ln\Lambda \;\approx\; -11.35 + 2\ln T_e - \frac{1}{2}2\ln p \qquad (T_e \text{ in °K, } p \text{ in atm}) \qquad (18)$$

so that $\ln\Lambda$ is usually around $6-12$, and can even be taken as a constant (~ 8) in rough calculations. For the neutral hydrogen atoms, the collisions are fairly "hard", and one can use the approximation,

$$Q_{eH} \;\approx\; 2\times 10^{-19}\ \text{m}^2 \qquad (19)$$